Representation of $PG(3,5)$ by $r$-Subspaces, $r = 1, 2$

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Abstract

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ABSTRACT

In this paper, the representation of projective space of dimension three has been studied by its subspaces of dimension one and two. Some new results have been found. And as special case, the projective space of dimension three over the Galois field $GF(5)$ has been partition into its lines and represented by its planes.

Key words: Finite projective space.

INTRODUCTION

The problem of partitions of the larger combinatorial into copies of smaller ones has a long history. These partitions might be based on objects of a given type like projective subspaces (Yff, P., 1977), caps (Ebert, G.L., 1985) or based on combine objects of different types like caps with subspaces what generally call a mixed partition (Baker, R.D., et al., 1999; Cossidente, A., 2000), or partition the projective space into Segre varieties (Bonisoli, A., et al., 1996; Bader, L., et al., 2002).

The aim of this paper is to discuss the partition of projective space, $PG(3,q)$ into 1–spaces (lines) and represented the space by 2–spaces (planes). As a special case the projective space, $PG(3,5)$ has been studied.

According to $PG(3,5)$, there are many mathematician studied this space like, Al-Mukhtar in 2008, surfaces and complete arcs were studied (Al-Mukhtar, A.SH., 2008). Also, in 2013, Topalova and Zhelezova, have been study the partition of the set of lines by spreads (Topalova, S., S. Zhelezova, 2013).

2- Notation and Background:

\[ F_q \]
\[ PG(n, q) \]
\[ \theta(n) = \frac{q^{n+1} - 1}{q-1} \]
\[ \vartheta(n) = \frac{(q^{n+1} - 1)(q^n - 1)}{(q^2 - 1)(q - 1)} \]
\[ \Theta(n) = \frac{q^n - 1}{q - 1} \]
\[ a \equiv b \mod m \]
\[ (m_1, m_2) \]
\[ M(T) \]
\[ C(F) \]

Galois Field of Order $q$

$n$-dimensional Projective Space Over $F_q$

Number of Points in $PG(n, q)$

Number of Lines in $PG(n, q)$

Number of Lines Through a Point

Mean \( a - b = m \)

Greatest Common Divisor of \( m_1, m_2 \)

Projectivity with \((n + 1) \times (n + 1)\) Matrix $T$

Companion Matrix of Polynomial $F$
Definition 1 (Hirschfeld, J.W.P., 1998):
A spread $F$ of $PG(n,q)$ by $r$-spaces is a set of $r$-spaces which partitions $PG(n,q)$ that is, every point of $PG(n,q)$ lies in exactly one $r$-space of $F$. Hence any two $r$-spaces of $F$ are disjoint.

Definition 2 (Hirschfeld, J.W.P., 1998):
A projectivity $r$ which permutes the $\theta(n)$ points of $PG(n,q)$ in a single cycle is called a cyclic projectivity (Singer cycle) and the group it generates a Singer group.

It is well known that every projective space has a cyclic projectivity $M(T)$, and to get the matrix $T$, it is enough to have a companion matrix $T = C(F)$ with subprimitive (primitive) characteristic polynomial $F$. Therefore, the points $P_i$ of $PG(n,q)$ can be construct as follows

$$P_i = U_0 \cdot T^i, \quad i = 0, \ldots, \theta(n).$$

where $U_0 = [1,0,\ldots,0]$. For more details, see [9, Chapter 4].

(i) Each $m$-space of $PG(n,q)$ occurs as one of a cycle of order $N$, where $N$ is some integer dividing $\theta(n)$. [Corollary 4.11].

(ii) If an $m$-space is one of a cycle of order $N$, its points can be written in the following form, modulo $\theta(n)$.

$$c_0 \ c_0 + N \ c_0 + (r - 2)N \ c_0 + (r - 1)N$$

where $\theta(n) = rN$, and $\theta(l) = r$ for some integer $l$. Also, a necessary condition that $N < \theta(n)$ is that

$$\theta(m), \theta(n)) > 1 \text{ or equivalently } (m + 1, n + 1) > 1.$$

[ Lemma 4.13, Lemma 4.15].

(iii) A spread of $m$-spaces exists in $PG(n,q)$ if and only if $m + 1$ divides $n + 1$. [Corollary 4.17].

(iv) The projective geometry, $PG(n,q)$ can be represented by an $n$-dimensional array which has $\prod_{k=1}^n \theta(i)$ entries.

If $(m + 1, n + 1) > 1, m \neq n, then

(i) there exists an $m$-space of cycle $N$ less than $\theta(n)$ such that, $\theta(n) = rN$, $r = \theta(1)$, $r$ divides $(\theta(m), \theta(n))$ and $l + 1$ divides $(m + 1, n + 1)$.

(ii) the points represented by $0, N, \ldots, (r - 1)N$ lie in an $l$-space.

As a direct result of Theorem 4, the following hold.

Corollary 5:
If $n = 1$ or 2, then there is no $m$-space of cycle $N$ less than $\theta(n)$.

3-New Results in $PG(3, q)$:

Let

$$M = \frac{\theta(3) - N}{\theta(3)}$$

denote the number of cycle of length $\theta(3)$.

Theorem 6:
In $PG(3, q)$,

(i) there is an 1-space of cycle $N$ less than $\theta(3)$;

(ii) the parameters in Theorem 4 are: $l = 1, r = q + 1$ and $N = q^2 + 1$;

(iii) there are $q$ cycle of length $\theta(3)$; that is, $M = q$ ;

(iv) the $\theta(2)$ lines of each plane divided as follows: one belong to a cycle of length $N$ and every $q + 1$ of the others to a cycle of length $\theta(3)$.

Proof:
(i) From Theorem 4.i.

(ii) Since $l + 1 = (m + 1, n + 1), so, l + 1 = 2$. Therefore; $l = 1$ and $r = \theta(1) = q + 1$.

$$N = \frac{\theta(2)}{r} = \frac{q^{l-1}}{q+1} = \frac{q^{l-1}(q^2+1)}{(q^2+1)} = q^2 + 1.$$

(iii) $$\frac{\theta(2)-N}{\theta(3)} = \frac{(q^3+1)(q^2+1)}{(q^3+1)(q^2+1)} = \frac{q(q^2+1)(q+1)}{(q^2-1)(q^2+1)} = \frac{q(q^2+1)(q+1)}{(q^2-1)(q^2+1)} = q.$$
Every \( q + 1 \) lines from the \( i \)-th cycle of order \( \theta(3) \) has \( q(q + 1) \) points in common in comes from piecewise intersection of these lines. The number of points out these intersections is

\[
(q + 1)(q + 1) - q(q + 1) = q + 1.
\]

These points form a line belongs to another cycle of order \( \theta(3) \). Thus, the numbers of lines which are belong to \( i \)-th cycle of order \( \theta(3) \) is \( q(q + 1) \). It is clear, to cover all points on \( PG(3, q) \), must each plane has at least one line belong to class of cycle \( N \).

Let \( L^i \) = Set of all lines in the \( i \)-th cycle, \( L^M \) of order \( \theta(3) \), \( L^{M+1} \) = Set of all lines in the \((M + 1)\)-th cycle of order \( N \), and \( L \) = Set of all lines in \( PG(3, q) \); that is,

\[
L = \bigcup_{i=1}^{M+1} L^i.
\]

Let \( \tau = M(T) \) be a cyclic projectivity over \( PG(3, q) \), where \( T \) is \( 4 \times 4 \) matrix has subprimitive characteristic polynomial.

Let \( \ell^i_1 = \ell^i T^{i-1}, \) where \( 1 \leq j \leq M \theta(3) + N \), and

\[
i = \begin{cases}
1, 2, ..., M & \text{if } (i - 1) \theta(3) + 1 \leq j \leq i \theta(3), \\
M + 1 & \text{if } M \theta(3) + 1 \leq j \leq M \theta(3) + N,
\end{cases}
\]

be the lines of \( PG(3, q) \).

Remark 7:

The number of cardinality of \( L^i \) is \( \theta(n) \) if \( i = 1, 2, ..., M \) and is \( N \) if \( i = M + 1 \).

From (1) the following three functions can be defined.

(i) For fixed \( i \), \( \lambda^i: [1, \theta(3)] \rightarrow L^i \);

\[
\lambda^i(k) = \ell^i T^k = \begin{cases}
\ell^i T^{(j+k-1) \mod \theta(3)} = \ell^i (j+k) \mod \theta(3), & \text{if } i = 1, 2, ..., M, \\
\ell^i T^{(j+k-1) \mod N} = \ell^i (j+k) \mod N, & \text{if } i = M + 1
\end{cases}
\]

(ii) For fixed \( \lambda_k: \mathcal{L} \rightarrow \mathcal{L} \);

\[
\lambda_k(\ell^i) = \ell^i T^{j+k-1} = \begin{cases}
\ell^{i(j+k-1) \mod \theta(3)} & \text{if } j + k \leq i \theta(3), \quad 1 \leq i \leq M \\
\ell^{i(M+1)(j+k-1) \mod N} = \ell^{i \theta(n)}(j+k) \mod N & \text{if } M \theta(3) < j + k \leq (i + 1) \theta(3), \quad 1 \leq i \leq M - 1
\end{cases}
\]

where, \( k = 1, ..., \theta(3) \).

(iii) For fixed \( i \) and \( k \), \( \lambda_k^i: \mathcal{L} \rightarrow \mathcal{L} \);

\[
\lambda_k^i(\ell^i) = \begin{cases}
\ell^i T^k = \ell^i (j+k) \mod \theta(3), & \text{if } 1 \leq i \leq M \\
\ell^{(M+1)j+k} = \ell^{(M+1)}(j+k) \mod N, & \text{if } i = M + 1
\end{cases}
\]

where \( k = 1, ..., \theta(3) \).

(iv) Let \( M = \{P_i | 0 \leq i \leq \theta(n) - 1, \ P_i \ \text{hyperplanes in} \ PG(n, q) \} \).

Define the function \( \zeta_k \) on \( M \) as follows:

\[
\zeta_k: M \rightarrow M; \ zeta_k(P_i) = P_i T^{j+k} = P_{(j+k) \mod \theta(n)}
\]

where \( k = 1, ..., \theta(n) \).

Theorem 8:

Over \( PG(3, q) \), the four functions \( \lambda^i, \lambda_k, \lambda_k^i, \zeta_k \) are bijective.

Remark 9:

The function \( \zeta_k \) is also bijective over any \( n \)-dimensional projective space.

4. Representation of \( PG(3,5) \) by \( r \)-Subspaces, \( r = 1, 2 \):

Let \( e \) be the primitive elements of \( GF(5) \). The following polynomial

\[
F(X) = X^4 - X^2 + X + e,
\]

is primitive polynomial over \( GF(5) \) of degree four. Then the projectivity \( \tau = M(T) \) given by the companion
is cyclic projectivity. The points of $PG(3,5)$ has the following form

$$P_i = P(0) \ast T^i, \quad i = 0, \ldots, 155.$$ 

The points, $P_0 = [1,0,0,0]$, $P_1 = [0,1,0,0]$, $P_2 = [0,0,1,0]$, $P_3 = [0,0,0,1]$. $P_{101} = [1,1,1,1]$ are the standard frame in $PG(3,5)$.

If $X, Y$ are two points in $PG(3,5)$, then the points of the line joining $X$ and $Y$ are those which can be expressed

$$aX + bY,$$

where $a, b \in F_5$ not both are zero. These lines have been indexed as given in (1).

Let $u_3$ represent the plane with fourth coordinate is zero. So, $u_3 = \{0, 1, 2, 4, 13, 20, 23, 24, 29, 31, 34, 38, 41, 44, 46, 58, 72, 73, 77, 88, 89, 95, 97, 98, 111, 120, 124, 139, 144, 150, 152\}$.

Therefore, to construct all others 155 planes, the cyclic projectivity used as follows:

$$P_i = u_3 \ast T^i, \quad i = 0, \ldots, \theta(3) - 1.$$ 

### Theorem 10:

Over $PG(3,5)$,

(i) there are five cycle of length $\theta(5)$;

(ii) there is one cycle of length 26;

(iii) the cycle $L^i$ is spread of 1-spaces in $PG(3,5)$.

**Proof:**

Let $\ell^1 = \{0, 1, 23, 72, 88, 97\}$, $\ell^2 = \{0, 2, 29, 44, 95, 150\}$, $\ell^3 = \{0, 3, 57, 119, 138, 149\}$, $\ell^4 = \{0, 4, 24, 38, 77, 124\}$, $\ell^5 = \{0, 5, 17, 48, 115, 128\}$, $\ell^6 = \{0, 26, 52, 78, 104, 130\}$.

then,

(i) $L^i = \{\ell^l T^j \mid 1 \leq j \leq \theta(5)\}, \quad \ell^i = 1, \ldots, 5$ are cycle of length $\theta(5)$.

(ii) $L^6 = \{\ell^6 T^j \mid 1 \leq j \leq 26\}$ is cycle of length 26.

(iii) Using the function $\lambda^i$, $\theta(5) + 1 \leq k \leq 5\theta(5) + 26$ the spread of 1-spaces in $PG(3,5)$ has founded as given bellow.

<table>
<thead>
<tr>
<th>$\ell_{823}$</th>
<th>0</th>
<th>26</th>
<th>52</th>
<th>78</th>
<th>104</th>
<th>130</th>
</tr>
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<td>53</td>
<td>79</td>
<td>105</td>
<td>131</td>
</tr>
<tr>
<td>$\ell_{806}$</td>
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<td>77</td>
<td>103</td>
<td>129</td>
<td>155</td>
</tr>
</tbody>
</table>
Table 1: Partition the projective plane $P_n$ into its lines $L^i$

<table>
<thead>
<tr>
<th>$L^i$</th>
<th>0</th>
<th>1</th>
<th>23</th>
<th>72</th>
<th>88</th>
<th>97</th>
</tr>
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<td>2</td>
<td>24</td>
<td>73</td>
<td>89</td>
<td>98</td>
</tr>
<tr>
<td>$L^2$</td>
<td>23</td>
<td>24</td>
<td>46</td>
<td>95</td>
<td>111</td>
<td>120</td>
</tr>
<tr>
<td>$L^3$</td>
<td>72</td>
<td>73</td>
<td>95</td>
<td>144</td>
<td>4</td>
<td>13</td>
</tr>
<tr>
<td>$L^4$</td>
<td>88</td>
<td>89</td>
<td>111</td>
<td>4</td>
<td>20</td>
<td>29</td>
</tr>
<tr>
<td>$L^5$</td>
<td>97</td>
<td>98</td>
<td>120</td>
<td>13</td>
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<td>38</td>
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<td>$L^6$</td>
<td>0</td>
<td>2</td>
<td>29</td>
<td>44</td>
<td>95</td>
<td>150</td>
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<tr>
<td>$L^7$</td>
<td>2</td>
<td>4</td>
<td>31</td>
<td>46</td>
<td>97</td>
<td>152</td>
</tr>
<tr>
<td>$L^8$</td>
<td>29</td>
<td>31</td>
<td>58</td>
<td>73</td>
<td>124</td>
<td>23</td>
</tr>
<tr>
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<td>73</td>
<td>88</td>
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<td>38</td>
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<td>73</td>
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<td>$L^{24}$</td>
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</tr>
</tbody>
</table>

**Theorem 11:**

(i) The planes of $PG(3,5)$ can be represented by its lines such that every six of them from different $i$th-cycle $L^i$ of order $\theta(3)$ and one from $L^6$.

(ii) The projective geometry, $PG(3,5)$ can be represented by a 3-dimensional array has $6 \cdot 31 \cdot 156$ entries, consist of 156 faces from its planes, each face consist of 31 lines and each line consist of 6 points.

**Proof:**

(i) This is a corollary to Theorem 6.iv, and as given in Table 1.

(ii) This is a corollary to Lemma 3.iv. Using the bijective map $\lambda^i$ in Remark 7, each row of Table 1. Transform to other the row using $k$. So, every plane starting with $P_0$ will transform to the other $P_i$ by $k$, where $i = 1, 2, ..., \theta(3)$ and $k = 1, 2, ..., \theta(3)$.

**REFERENCES**


